Ex: Prove $n < 2^n$ for all $n \geq 0$.

Sol: We proceed by induction on n.

Base Case: When $n=0$, we have $n = 0 < 1 = 2^0 = 2^n$.

When $n=1$, we have $n = 1 < 2 = 2^1 = 2^n$.

Hence the base cases holds.

~~$n < 2^n$~~ ~~$0 < 2^0$~~ by themselves not enough

Inductive Step: Suppose $n < 2^n$ for some $n \geq 1$. Now we compute

WTS: $\forall n\,[P(n) \to P(n+1)]$

WTS: $P(n+1)$ i.e. $n+1 < 2^{n+1}$

$$\begin{aligned} n+1 &< 2^n + 1 && \text{(by IH)} \\ &\leq 2^n + n && \text{(by } n \geq 1\text{)} \\ &< 2^n + 2^n && \text{(by IH)} \\ &= 2(2^n) = 2^{n+1} && \text{(by basic arithmetic)}. \end{aligned}$$

Hence $n+1 < 2^{n+1}$ and we have verified the inductive step.

Hence we conclude the original statement is true by weak mathematical induction. ☐

$$A \leftrightarrow B \overset{?}{\equiv} (A \to B) \wedge ((\neg A) \to (\neg B)) \quad \checkmark$$

$$\begin{aligned} A \leftrightarrow B &\equiv (A \to B) \wedge (B \to A) && \text{(Bicond. Exp.)} \\ &\equiv (A \to B) \wedge ((\neg A) \to (\neg B)) && \text{(Contrapositive)} \\ &\equiv ((\neg B) \to (\neg A)) \wedge ((\neg A) \to (\neg B)) && \text{(Contrapositive)} \\ &\equiv (\neg B) \leftrightarrow (\neg A) && \text{(Bicond. Exp.)}. \end{aligned}$$

$$P(0) \wedge \forall_{k\geq 0}\left[P(k) \to P(k+1)\right] \equiv \forall_{n\geq 0}\left[P(n)\right]$$

We must show $\sum_{k=1}^{n+1} k = \frac{1}{2}(n+1)(n+2)$. ← Good form

Scrap work

$$\sum_{k=1}^{n+1} k \overset{WTS}{=} \frac{1}{2}(n+1)(n+2)$$

$$= \sum_{k=1}^{n} k + (n+1) \overset{WTS}{=} \frac{1}{2}(n+1)(n+2)$$

$$= \frac{1}{2}n(n+1) + (n+1) \overset{WTS}{=} \frac{1}{2}(n+1)(n+2)$$

$$= (n+1)\left(\frac{1}{2}n + 1\right) \overset{WTS}{=} \frac{1}{2}(n+1)(n+2)$$

$$= \frac{1}{2}(n+1)(n+2) \overset{WS}{=} \frac{1}{2}(n+1)(n+2)$$ ✓

Each is a statement...

BAD Form

Ex: Prove $A \cup B = B \cup A$. ← ★

("Element Method")

Sol 1: Let A and B be sets.

($\subseteq$): Let $x \in A \cup B$. Thus either $x \in A$ or $x \in B$ by definition of set union. Hence either $x \in B$ or $x \in A$ by commutativity of disjunction. Finally, $x \in B \cup A$ by definition of set union. Hence as x was

arbitrary, $A \cup B \subseteq B \cup A$ as desired.

($\supseteq$): (left as an exercise).

Hence $A \cup B \subseteq B \cup A$ and $B \cup A \subseteq A \cup B$, so $A \cup B = B \cup A$. ☑

Sol 2: Let A and B be sets. We have the following:

$$\begin{aligned} A \cup B &= \{x : x \in A \text{ or } x \in B\} && \text{(by definition of set union)} \\ &= \{x : x \in B \text{ or } x \in A\} && \text{(by commutativity of } \vee\text{)} \\ &= B \cup A. && \text{(by definition of union)} \end{aligned}$$

Hence $A \cup B = B \cup A$ as desired. ☑